

# On three duality results

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The aim of this short note is to give counterexamples to two results by D. Y. Gao [5, Th. 16], [4, Th. 2] and to improve a related result by S.-C. Fang, D. Y. Gao, R.-L. Sheu and S.-Y. Wu [1, Th. 3].

## 1 Counterexamples to [5, Th. 16], [4, Th. 2]

On [5, page 298] the authors consider the problem

$$\min \{ P(x) = \frac{1}{2}x^T A x - f^T x : \frac{1}{2}x^T C x \leq \lambda, x \in \mathbb{R}^n \}. \quad (8.156)''$$

“...where  $A$  and  $C$  are two symmetrical matrices in  $\mathbb{R}^{n \times n}$ ,  $f \in \mathbb{R}^n$  is a given vector, and  $\lambda \in \mathbb{R}$  is a given constant”, and continue on the following page with: “On the dual feasible space

$$\mathcal{V}_k^* = \{ \varsigma \in \mathbb{R} \mid \varsigma \geq 0, \det(A + \varsigma C) \neq 0 \}$$

and the canonical dual problem (8.155) can be formulated as (see [50]):

$$\max \{ P^d(\varsigma) = -\frac{1}{2}f^T(A + \varsigma C)^{-1}f - \lambda \varsigma : \varsigma \in \mathcal{V}_k^* \}. \quad (8.158)''$$

“The following result was obtained recently.

**Theorem 16 (Gao [50])** Suppose that the matrix  $C$  is positive definite, and  $\bar{\varsigma} \in \mathcal{V}_a^*$  is a critical point of  $P^d(\varsigma)$ . If  $A + \bar{\varsigma}C$  is positive definite, the vector

$$\bar{x} = (A + \bar{\varsigma}C)^{-1}f$$

is a global minimizer of the primal problem (8.156). However, if  $A + \bar{\varsigma}C$  is negative definite, the vector  $\bar{x} = (A + \bar{\varsigma}C)^{-1}f$  is a local minimizer of the primal problem (8.156).”

In the previous statement  $\mathcal{V}_a^* = [0, +\infty)$  (see [5, p. 297]) while reference [50] is our reference [3]. This first result we are interested in is cited in [5] as being published in [3]; however, we could not find its statement in [3]. The following is a counterexample for [5, Th. 16].

**Example 1** Consider

$$A = \begin{bmatrix} -2 & -1 \\ -1 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad f = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \lambda = \frac{1}{2}.$$

Then  $P^d(y) = -\frac{1}{2}y - \frac{1}{2} \frac{2y-3}{y^2-5y+5}$  and  $(P^d)'(y) = -\frac{1}{2} \frac{(y-2)^2}{(y^2-5y+5)^2} (y-1)(y-5)$ . Hence the set of critical points of  $P^d$  is  $\{1, 2, 5\}$  all contained in  $\mathcal{V}_a^*$ . For  $\bar{y} = 1$  we have that  $A + \bar{y}C = \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}$  is negative definite and  $\bar{x} = (A + \bar{y}C)^{-1}f = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ .

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Let  $\mathcal{U}_0 := \{(\cos t, \sin t)^T \mid t \in (-\pi, \pi)\}$  be a subset of the admissible set  $\mathcal{U} = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$  and

$$f(t) := P((\cos t, \sin t)^T) = -1 - \cos t \sin t - \frac{1}{2} \sin^2 t + \cos t + \sin t = -(3 + \cos t - 2 \sin t) \sin^2 \frac{1}{2} t,$$

$t \in \mathbb{R}$ ; hence  $P(\bar{x}) = f(0) = 0 = P^d(\bar{y})$ . According to the previous theorem  $\bar{x}$  should be local minimizer of  $P$  on  $\mathcal{U}$ , in contradiction to the fact that  $t = 0$  is a strict local maximum point of  $f$  (see Figure 1).

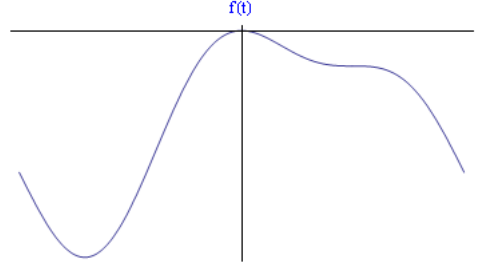


Figure 1.

Our attention turns to the problem considered in [4]

$$(\mathcal{P}) : \min \{P(x) = U(\Lambda(x)) + Q(x) : x \in \mathcal{R}^n\} \quad (5)$$

where “ $Q(x) = \frac{1}{2}x^T A x - c^T x$  is a quadratic function,  $A = A^T \in \mathcal{R}^{n \times n}$  is a given symmetric matrix”,  $c \in \mathcal{R}^n$ , and the so called “geometrical operator  $\Lambda : \mathcal{R}^n \rightarrow \mathcal{R}^{1+n}$  and the associated canonical function  $U$  can be introduced as following:

$$y = \Lambda(x) = \begin{pmatrix} \xi(x) \\ \epsilon(x) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}|Bx|^2 - \alpha \\ \{x_i^2 - \ell_i\} \end{pmatrix} \in \mathcal{R}^{1+n},$$

$$U(y) = \frac{1}{2}\xi^2 + \Psi(\epsilon) \quad (3)$$

where

$$\Psi(\epsilon) = \begin{cases} 0 & \text{if } \epsilon \leq 0, \\ +\infty & \text{otherwise.} \end{cases} \quad (4)$$

Here “ $B \in \mathcal{R}^{m \times n}$  is a given matrix and  $\alpha > 0$  is a given parameter” while  $\ell = \{\ell_i\} \in \mathcal{R}^n$ ,  $\ell_i \geq 0$ . “The notation  $|x|$  used in this paper denotes the Euclidean norm of  $x$ ”.

“The canonical dual problem of  $(\mathcal{P})$  can be proposed as the following

$$(\mathcal{P}^d) : \text{sta} \left\{ P^d(\varsigma, \sigma) = -\frac{1}{2}c^T [G(\varsigma, \sigma)]^{-1} c - \frac{1}{2}\varsigma^2 - \alpha\varsigma - \ell^T \sigma : (\varsigma, \sigma)^T \in \mathcal{S}_a \right\}. \quad (11)$$

Here “ $G(\varsigma, \sigma)$  is a symmetrical matrix, defined by

$$G(\varsigma, \sigma) = A + \varsigma B^T B + 2 \text{Diag}(\sigma) \in \mathcal{R}^{n \times n}, \quad (9)$$

and  $\text{Diag}(\sigma) \in \mathcal{R}^{n \times n}$  denotes a diagonal matrix with  $\{\sigma_i\}$  ( $i = 1, 2, \dots, n$ ) as its diagonal

entries” while “ $\mathcal{S}_a = \{y^* = \begin{pmatrix} \varsigma \\ \sigma \end{pmatrix} \in \mathcal{R}^{1+n} \mid \varsigma \geq -\alpha, \sigma \geq 0, \det G(\varsigma, \sigma) \neq 0\}$ . (10)”

One continues with “we need to introduce some useful feasible spaces:

$$\mathcal{S}_a^+ = \{(\varsigma, \sigma)^T \in \mathcal{S}_a \mid G(\varsigma, \sigma) \text{ is positive definite}\}, \quad (16)$$

$$\mathcal{S}_a^- = \{(\varsigma, \sigma)^T \in \mathcal{S}_a \mid G(\varsigma, \sigma) \text{ is negative definite}\}. \quad (17)$$

**Theorem 2** (Triality Theorem). Suppose that the vector  $\bar{y}^* = (\bar{\varsigma}, \bar{\sigma})^T$  is a KKT point of the canonical dual function  $P^d(y^*)$  and  $\bar{x} = [G(\bar{\varsigma}, \bar{\sigma})]^{-1}c$ .

If  $\bar{y}^* = (\bar{\varsigma}, \bar{\sigma})^T \in \mathcal{S}_a^+$ , then  $\bar{y}^*$  is a global maximizer of  $P^d$  on  $\mathcal{S}_a^+$ , the vector  $\bar{x}$  is a global minimizer of  $P$  on  $\mathcal{X}_a$ , and

$$P(\bar{x}) = \min_{x \in \mathcal{X}_a} P(x) = \max_{y^* \in \mathcal{S}_a^+} P^d(y^*) = P^d(\bar{y}^*). \quad (18)$$

If  $\bar{y}^* \in \mathcal{S}_a^-$ , on the neighborhood  $\mathcal{X}_o \times \mathcal{S}_o \subset \mathcal{X}_a \times \mathcal{S}_a$  of  $(\bar{x}, \bar{y}^*)$ , we have that either

$$P(\bar{x}) = \min_{x \in \mathcal{X}_o} P(x) = \min_{y^* \in \mathcal{S}_o} P^d(y^*) = P^d(\bar{y}^*) \quad (19)$$

holds, or

$$P(\bar{x}) = \max_{x \in \mathcal{X}_o} P(x) = \max_{y^* \in \mathcal{S}_o} P^d(y^*) = P^d(\bar{y}^*). \quad (20)$$

Recall that “ $\mathcal{X}_a = \{x \in \mathcal{R}^n \mid \ell^l \leq x \leq \ell^u\}$  is a feasible space” and “we assume without loss of generality that  $\ell^u = -\ell^l = \ell^{\frac{1}{2}} = \{\sqrt{\ell_i}\}$  (if necessary, a simple linear transformation can be used to convert the problem to this form).”

A few remarks are necessary at this moment.

- Note that in [4, Th. 2] the meaning of “ $\bar{y}^* = (\bar{\varsigma}, \bar{\sigma})^T$  is a KKT point of  $(\mathcal{P}^d)$ ” is not explained. However, due to the fact that the constraints of problem  $(\mathcal{P}^d)$  are expressed via  $\mathcal{S}_a$ , if  $\bar{y}^* \in \text{int } \mathcal{S}_a$  is a critical point of  $P^d$  (that is,  $\nabla P^d(\bar{y}^*) = 0$ ) then  $\bar{y}^*$  is a KKT point.
- It is not clear whether the neighborhood  $\mathcal{X}_o \times \mathcal{S}_o$  is “a priori” prescribed or the statement should be understood in the sense that there exists such a neighborhood. In any case the example below shows that [4, Th. 2] is false. The proof of this Triality Theorem in [4] begins with “In the canonical form of the primal problem (5), replacing  $U(\Lambda(y))$  by the Fenchel-Young equality  $(\Lambda(x))^T y^* - U^\natural(y^*)$ , the Gao-Strang type *total complementary function* (see [22]) associated with  $(\mathcal{P})$  can be obtained as  $\Xi(x, y^*) = \frac{1}{2}x^T G(\varsigma, \sigma)x - U^\natural(y^*) - x^T c - \alpha \varsigma - \ell^T \sigma$ . (21)”. For the proof of the second part of the theorem one says: “On the other hand, if  $\bar{y}^* \in \mathcal{S}_a^-$ , the matrix  $G(\bar{\varsigma}, \bar{\sigma})$  is negative definite. In this case, the total complementary function  $\Xi(x, y^*)$  defined by (21) is a so-called super-Lagrangian (see [12]), i.e., it is locally concave in both  $x \in \mathcal{X}_o \subset \mathcal{X}_a$  and  $y^* \in \mathcal{S}_o \subset \mathcal{S}_a$ . Thus, by the triality theory developed in [12], we have either

$$P(\bar{x}) = \min_{x \in \mathcal{X}_o} P(x) = \min_{x \in \mathcal{X}_o} \max_{y^* \in \mathcal{S}_o} \Xi(x, \lambda) = \min_{y^* \in \mathcal{S}_o} \max_{x \in \mathcal{X}_o} \Xi(x, \lambda) = \min_{y^* \in \mathcal{S}_o} P^d(y^*),$$

or

$$P(\bar{x}) = \max_{x \in \mathcal{X}_o} P(x) = \max_{x \in \mathcal{X}_o} \max_{y^* \in \mathcal{S}_o} \Xi(x, \lambda) = \max_{y^* \in \mathcal{S}_o} \max_{x \in \mathcal{X}_o} \Xi(x, \lambda) = \max_{y^* \in \mathcal{S}_o} P^d(y^*).$$

This proves the statements (19) and (20).”

The references [22] and [12] mentioned above are our references [6] and [2], respectively. Therefore the second part of the conclusion for [4, Th. 2] does not follow from a specific results with assumptions that can be verified but from “the triality theory”.

**Example 2** Let  $n = 2$ ,  $A = -4I_2$ ,  $B = I_2$ ,  $c = (-2, -2)^T$ ,  $\alpha = 3$ ,  $\ell = (4, 4)^T$ . We have that

$$P(s, t) = -2s^2 - 2t^2 + 2s + 2t + \frac{1}{2} \left( \frac{1}{2}s^2 + \frac{1}{2}t^2 - 3 \right)^2,$$

and the restrictions are  $s^2 \leq 4$ ,  $t^2 \leq 4$ , that is  $\mathcal{X}_a = [-2, 2]^2$ . Also,

$$P^d((y, \sigma, \tau)^T) = -\frac{2}{y-4+2\sigma} - \frac{2}{y-4+2\tau} - \frac{1}{2}y^2 - 3y - 4\sigma - 4\tau.$$

Then  $\bar{y}^* = (1, 1, 1)^T \in \text{int } \mathcal{S}_a$  and  $\bar{y}^* \in \mathcal{S}_a^-$  since  $G((1, 1, 1)^T) = -I_2$ ,  $\bar{y}^*$  is a KKT point of  $P^d$  because  $\nabla P^d((1, 1, 1)^T) = 0$  and  $\bar{y}^* \in \text{int } \mathcal{S}_a$ , and  $\bar{x} = [G((1, 1, 1)^T)]^{-1}c = -c = (2, 2)^T \in$

$\mathcal{X}_a$ . Note that  $P(\bar{x}) = P^d((1, 1, 1)^T) = -15/2$ . On one hand, for  $\gamma \in (0, 1)$  we have that  $(2 - \gamma, 2 - \gamma)^T \in \mathcal{X}_a$  and

$$P((2 - \gamma, 2 - \gamma)^T) = -\frac{15}{2} + \frac{1}{2}\gamma^4 - 4\gamma^3 + 5\gamma^2 + 8\gamma > P(\bar{x}),$$

which shows that  $\bar{x}$  is not a maximum point of  $P$  on any neighborhood of  $\bar{x} \in \mathcal{X}_a$ . Hence relation (20) in the above theorem does not hold.

On the other hand, for  $\gamma \in (0, 1)$  we have that

$$P^d((1 - 16\gamma, 1 + 7\gamma, 1 + 7\gamma)^T) = -\frac{15}{2} - 16\frac{\gamma^2}{2\gamma + 1}(16\gamma + 7) < P^d((1, 1, 1)^T),$$

which shows that  $\bar{y}^* \in \text{int } \mathcal{S}_a$  is not a local minimum point of  $P^d$ . Hence relation (19) in the above theorem does not hold, too. Therefore, [4, Th. 2] is false.

## 2 On a theorem in [1]

Reference [1] begins with: “In this paper, we consider a simple 0-1 quadratic programming problem in the following form:

$$(\mathcal{P}) : \min / \max \{P(x) = \frac{1}{2}x^T Qx - f^T x \mid x \in \mathcal{X}_a\}, \quad (1)$$

where  $x$  and  $f$  are real  $n$ -vectors,  $Q \in \mathbb{R}^{n \times n}$  is a symmetrical matrix of order  $n$  and

$$\mathcal{X}_a = \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1, i = 1, 2, \dots, n\} \cap \mathcal{I}^n. \quad (2)$$

with  $\mathcal{I}^n = \{x \in \mathbb{R}^n \mid x_i \text{ is an integer, } i = 1, 2, \dots, n\}$ , continued with “By the definition of  $\Lambda(x)$  and  $V^1(\sigma)$ , we have

$$\Xi(x, \sigma) = \frac{1}{2}x^T Q_d(\sigma)x - x^T(f + \sigma), \quad (8)$$

where

$$Q_d(\sigma) = Q + 2 \text{Diag}(\sigma)$$

and  $\text{Diag}(\sigma) \in \mathbb{R}^{n \times n}$  is a diagonal matrix with  $\sigma_i, i = 1, 2, \dots, n$ , being its diagonal elements” and

$$“P^d(\sigma) = -\frac{1}{2}(f + \sigma)^T Q_d^{-1}(\sigma)(f + \sigma). \quad (9)”$$

Moreover, “we introduce the following four sets for consideration:

$$\mathcal{S}_\#^+ = \{\sigma \in \mathbb{R}^n \mid \sigma > 0, Q_d(\sigma) \text{ is positive definite}\}, \quad (22)$$

$$\mathcal{S}_\#^- = \{\sigma \in \mathbb{R}^n \mid \sigma > 0, Q_d(\sigma) \text{ is negative definite}\}, \quad (23)”$$

(we omit the other two sets).

“Then we have the following result on the global and local optimality conditions:

**Theorem 3.** Let  $Q$  be a symmetric matrix and  $f \in \mathbb{R}^n$ . Assume that  $\bar{\sigma}$  is critical point of  $P^d(\sigma)$  and  $\bar{x} = [Q_d(\bar{\sigma})]^{-1}(f + \bar{\sigma})$ .

(a) If  $\bar{\sigma} \in \mathcal{S}_\#^+$ , then  $\bar{x}$  is a global minimizer of  $P(x)$  over  $\mathcal{X}_a$  and  $\bar{\sigma}$  is a global maximizer of  $P^d(\sigma)$  over  $\mathcal{S}_\#^+$  with

$$P(\bar{x}) = \min_{x \in \mathcal{X}_a} P(x) = \max_{\sigma \in \mathcal{S}_\#^+} P^d(\sigma) = P^d(\bar{\sigma}). \quad (26)$$

(b) If  $\bar{\sigma} \in \mathcal{S}_\#^-$ , then  $\bar{x}$  is a local minimizer of  $P(x)$  over  $\mathcal{X}_a$  if and only if  $\bar{\sigma}$  is a local minimizer of  $P^d(\sigma)$  over  $\mathcal{S}_\#^-$ , i.e., in a neighborhood  $\mathcal{X}_o \times \mathcal{S}_o \subset \mathcal{X}_a \times \mathcal{S}_\#^-$  of  $(\bar{x}, \bar{\sigma})$ ,

$$P(\bar{x}) = \min_{x \in \mathcal{X}_o} P(x) = \min_{\sigma \in \mathcal{S}_o} P^d(\sigma) = P^d(\bar{\sigma}). \quad (27)”$$

Note that because  $\mathcal{X}_a$  is a discrete set any  $x \in \mathcal{X}_a$  is a local minimum point for  $P$  on  $\mathcal{X}_a$ , as well as a local maximum point of  $P$ . In fact the following stronger statement is true.

**Theorem 3** Let  $Q$  be a symmetric matrix and  $f \in \mathbb{R}^n$ . Assume that  $\bar{\sigma}$  is critical point of  $P^d$  such that  $\det Q_d(\bar{\sigma}) \neq 0$ , and  $\bar{x} := [Q_d(\bar{\sigma})]^{-1}(f + \bar{\sigma})$ . Then  $\bar{x} \in \mathcal{X}_a$  and  $P(\bar{x}) = \Xi(\bar{x}, \bar{\sigma}) = P^d(\bar{\sigma})$ .

(a) If  $\bar{\sigma} \in \mathcal{S}_{\#}^+$ , then  $\bar{\sigma}$  is a global maximizer of  $P^d$  over  $\mathcal{S}_{\#}^+$  and  $\bar{x}$  is a global minimizer of  $P$  over  $\mathcal{X} := [0, 1]^n$ ; in particular,  $\bar{x}$  is a global minimizer of  $P$  over  $\mathcal{X}_a = \{0, 1\}^n$ .

(b) If  $\bar{\sigma} \in \mathcal{S}_{\#}^-$ , then  $\bar{x}$  is a local minimizer of  $P$  over  $\mathcal{X}$  and  $\bar{\sigma}$  is a global minimizer of  $P^d$  over  $\mathcal{S}_{\#}^-$ .

Note that the first part of the above theorem practically covers Theorems 1 and 2 in [1].

Proof. It is obvious that  $\Xi(x, \cdot)$  is affine (hence concave and convex) for every  $x \in \mathbb{R}^n$ ,  $\Xi(\cdot, \sigma)$  is convex for  $\sigma \in \mathcal{S}_{\#}^+$ , and  $\Xi(\cdot, \sigma)$  is concave for  $\sigma \in \mathcal{S}_{\#}^-$ . Note that

$$\nabla_x \Xi(x, \sigma) = Q_d(\sigma)x - (f + \sigma), \quad \nabla_{\sigma} \Xi(x, \sigma)(v) = x^T \text{Diag}(v)x - x^T v \quad \forall v \in \mathbb{R}^n; \quad (1)$$

it follows that  $\nabla_{\sigma} \Xi(x, \sigma) = 0$  if and only if  $x_i^2 - x_i = 0$  for every  $i \in \overline{1, n}$ , that is,  $x \in \mathcal{X}_a$ . Furthermore, due to the fact that a critical point of a convex function is a global minimum point, we have

$$P^d(\sigma) = \Xi([Q_d(\sigma)]^{-1}(f + \sigma), \sigma) = \begin{cases} \min_{x \in \mathbb{R}^n} \Xi(x, \sigma) & \text{if } \sigma \in \mathcal{S}_{\#}^+, \\ \max_{x \in \mathbb{R}^n} \Xi(x, \sigma) & \text{if } \sigma \in \mathcal{S}_{\#}^-. \end{cases} \quad (2)$$

Recall the fact that the operator  $\varphi : \{U \in \mathfrak{M}_n \mid U \text{ invertible}\} \rightarrow \mathfrak{M}_n$  defined by  $\varphi(U) = U^{-1}$  is Fréchet differentiable and  $d\varphi(U)(S) = -U^{-1}SU^{-1}$  for  $U, S \in \mathbb{R}^{n \times n}$  with  $U$  invertible, where  $\mathfrak{M}_n$  is the (normed) linear space of  $n \times n$  real matrices. Also, we have  $dQ_d(\sigma)(v) = 2 \text{Diag}(v)$  and so, on  $\mathcal{S}^a = \{\sigma \in \mathbb{R}^n \mid \det Q_d(\sigma) \neq 0\}$ ,  $d[Q_d(\sigma)]^{-1}(v) = -2[Q_d(\sigma)]^{-1} \text{Diag}(v)[Q_d(\sigma)]^{-1}$  and

$$\begin{aligned} dP^d(\sigma)(v) &= -v^T [Q_d(\sigma)]^{-1}(f + \sigma) + (f + \sigma)^T [Q_d(\sigma)]^{-1} \text{Diag}(v) [Q_d(\sigma)]^{-1}(f + \sigma), \quad (3) \\ d^2 P^d(\sigma)(v, v) &= -v^T [Q_d(\sigma)]^{-1} v + 4v^T [Q_d(\sigma)]^{-1} \text{Diag}(v) [Q_d(\sigma)]^{-1}(f + \sigma) \\ &\quad - 4(f + \sigma)^T [Q_d(\sigma)]^{-1} \text{Diag}(v) [Q_d(\sigma)]^{-1} \text{Diag}(v) [Q_d(\sigma)]^{-1}(f + \sigma) \end{aligned} \quad (4)$$

for all  $v \in \mathbb{R}^n$ .

Since  $\bar{\sigma} \in \mathcal{S}^a$  is a critical point of  $P^d$  we have that  $dP^d(\bar{\sigma}) = 0$ . Taking into account (1), we obtain from (3), using a direct computation, that  $\nabla_{\sigma} \Xi(\bar{x}, \bar{\sigma}) = 0$ , and so  $\bar{x} \in \mathcal{X}_a \subset \mathcal{X}$ .

Moreover, since  $x_i^2 = x_i$

$$\begin{aligned} P(\bar{x}) &= \frac{1}{2} \bar{x}^T Q \bar{x} - f^T \bar{x} = \frac{1}{2} \bar{x}^T Q_d(\bar{\sigma}) \bar{x} - \bar{x}^T f - \bar{x}^T \text{Diag}(\bar{\sigma}) \bar{x} \\ &= \frac{1}{2} \bar{x}^T Q_d(\bar{\sigma}) \bar{x} - \bar{x}^T f - \bar{x}^T \bar{\sigma} = \Xi(\bar{x}, \bar{\sigma}) \\ &= \frac{1}{2} (f + \bar{\sigma})^T [Q_d(\bar{\sigma})]^{-1} (f + \bar{\sigma}) - \bar{x}^T (f + \bar{\sigma}) = P^d(\bar{\sigma}). \end{aligned}$$

It is clear that  $\mathcal{S}_{\#}^+$  and  $\mathcal{S}_{\#}^-$  are open convex sets because  $\sigma \rightarrow Q_d(\sigma)$  is affine.

If  $A := [Q_d(\sigma)]^{-1}$  is positive definite, setting  $w := \text{Diag}(v) [Q_d(\sigma)]^{-1} (f + \sigma)$  we have for every  $v \in \mathbb{R}^n$  that

$$d^2 P^d(\sigma)(v, v) = -v^T A v + 4v^T A w - 4w^T A w = -(v - 2w)^T A (v - 2w) \leq 0,$$

i.e.  $d^2 P^d(\sigma)$  is seminegatively definite. Hence  $P^d$  is concave on  $\mathcal{S}_\#^+$ . Similarly,  $P^d$  is convex on  $\mathcal{S}_\#^-$ .

(a) Let  $\bar{\sigma} \in \mathcal{S}_\#^+$ . Since  $\Xi(x, \sigma) = P(x) + \sum_{i=1}^n \sigma_i(x_i^2 - x_i) \leq P(x)$ , for every  $\sigma \geq 0$ ,  $x \in [0, 1]^n$  and taking (2) into account we get

$$\begin{aligned} P^d(\bar{\sigma}) &\leq \sup_{\sigma \in \mathcal{S}_\#^+} P^d(\sigma) = \sup_{\sigma \in \mathcal{S}_\#^+} \min_{x \in \mathbb{R}^n} \Xi(x, \sigma) \leq \sup_{\sigma \geq 0} \inf_{x \in [0, 1]^n} \Xi(x, \sigma) \\ &\leq \inf_{x \in [0, 1]^n} \sup_{\sigma \geq 0} \Xi(x, \sigma) \leq \inf_{x \in [0, 1]^n} P(x) \leq P(\bar{x}). \end{aligned}$$

Therefore  $P^d(\bar{\sigma}) = \max_{\sigma \in \mathcal{S}_\#^+} P^d(\sigma)$  and  $P(\bar{x}) = \min_{x \in [0, 1]^n} P(x)$ , since  $P(\bar{x}) = P^d(\bar{\sigma})$ .

(b) Take now  $\bar{\sigma} \in \mathcal{S}_\#^-$ . Since  $P^d$  is convex on  $\mathcal{S}_\#^-$  and  $\bar{\sigma}$  is a critical point of  $P^d$ , clearly  $\bar{\sigma}$  is a global minimizer of  $P^d$  on  $\mathcal{S}_\#^-$ .

Consider  $x \in \mathcal{X}$ . Since  $Q\bar{x} - f = \bar{\sigma} - 2 \text{Diag}(\bar{\sigma})\bar{x}$ , we get

$$\begin{aligned} P(x) &= \frac{1}{2}x^T Qx - f^T x = \frac{1}{2}(x - \bar{x})^T Q(x - \bar{x}) + (x - \bar{x})^T Q\bar{x} + \frac{1}{2}\bar{x}^T Q\bar{x} - f^T x \\ &= P(\bar{x}) + \frac{1}{2}(x - \bar{x})^T Q(x - \bar{x}) + (x - \bar{x})^T Q\bar{x} - (x - \bar{x})^T f \\ &= P(\bar{x}) + \frac{1}{2}(x - \bar{x})^T Q(x - \bar{x}) + \sum_{i=1}^n \bar{\sigma}_i(1 - 2\bar{x}_i)(x_i - \bar{x}_i) = P(\bar{x}) + \sum_{i=1}^n \mu_i(x_i - \bar{x}_i), \end{aligned}$$

where  $Q = \{q_{ij}\}$  and  $\mu_i := \bar{\sigma}_i(1 - 2\bar{x}_i) + \frac{1}{2} \sum_{j=1}^n q_{ij}(x_j - \bar{x}_j)$ .

Let  $\varepsilon > 0$  be such that  $\min_{k \in \overline{1, n}} \bar{\sigma}_k \geq \frac{n}{2}\varepsilon \max_{i, j \in \overline{1, n}} |q_{ij}|$ . Take  $U = \{x \in \mathbb{R}^n \mid |x_i - \bar{x}_i| \leq \varepsilon \forall i \in \overline{1, n}\}$ . Then  $\frac{1}{2} \sum_{j=1}^n q_{ij}(x_j - \bar{x}_j)(x_i - \bar{x}_i) \geq -\frac{n}{2}\varepsilon |x_i - \bar{x}_i| \max_{i, j \in \overline{1, n}} |q_{ij}|$  for every  $i \in \overline{1, n}$  and  $x \in U$ , while the inequality  $\bar{\sigma}_i(1 - 2\bar{x}_i)(x_i - \bar{x}_i) \geq |x_i - \bar{x}_i| \min_{k \in \overline{1, n}} \bar{\sigma}_k$  for every  $i \in \overline{1, n}$  and  $x \in U \cap \mathcal{X}$  is easily checked, since  $\bar{x}_i \in \{0, 1\}$ . This shows that  $\mu_i(x_i - \bar{x}_i) \geq 0$  for every  $i \in \overline{1, n}$ , whence  $P(x) \geq P(\bar{x})$  for every  $x \in U \cap \mathcal{X}$ ; therefore,  $\bar{x}$  is a local minimizer of  $P$  on  $\mathcal{X}$ .

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